Systems of Relaxation Oscillators with Time-Delayed Coupling

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Abstract

We study systems of relaxation oscillators in the presence of nearest-neighbor coupling. It is shown that introduction of delay in the coupling leads to a slowing-down of the frequency of individual oscillators as well as of the average frequency of the system. In certain parameter ranges, we also observe the existence of metastable states. Both phenomena can be understood in a simple mean-field like theory.

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The recently observed synchronized oscillations in the visual cortex of the cat [3, 5, 9] may have a role as a link between events on the neuronal and the perceptive level [1, 4]. Although it is not clear at present if this is really the functional significance of the observed oscillations, this hypothesis is supported by psychophysical evidence [17]. The possible far-reaching consequences make a theoretical understanding of this phenomenon desirable. Unfortunately, the details of the elements of which the oscillating circuits are composed are not known. In light of this ignorance, we decided to make as few assumptions as possible and use nothing but the observed facts that (a) oscillations are present and (b) under suitable circumstances, oscillations are synchronized over large distances, even across the cerebral hemispheres [6].

We therefore consider a system consisting of elements with only two properties: (a) they show periodic behavior and (b) their interaction favors synchrony. An isolated element is described by the equation

$$\frac{d\phi_i(t)}{dt} = \omega_i \tag{1}$$

where ϕ_i is the state variable of oscillator i and ω_i is the intrinsic frequency of this oscillator. Adding a simple interaction term which favors synchrony, we obtain

$$\frac{d\phi_i(t)}{dt} = \omega_i + K \sum_j \sin(\phi_j(t) - \phi_i(t)) \tag{2}$$

where the sum runs over the subset of the N oscillators which are coupled to oscillator i. We have also introduced the coupling constant $K > 0$.

This and similar systems have been studied extensively over the last years [11, 8, 12, 7, 2, 13, 14, 16]. In this report, we will introduce a seemingly small change in eq. 2: Instead of assuming instantaneous interaction between coupled oscillators, we assume that it takes some time τ until the changed state of oscillator j can affect oscillator i:

$$\frac{d\phi_i(t)}{dt} = \omega_i + K \sum_j \sin(\phi_j(t-\tau) - \phi_i(t)) \qquad \tau \geq 0 \tag{3}$$

It is obvious that this is a more realistic model for interactions in a nervous system than eq. 2: interaction between neurons is always subject to a delay due to the finite conduction velocity of neural signals along the nerve fibers. In the case of a chemical synapse, the delay is even larger due to synaptic delay.

In fact, equation 4 is a more realistic model than eq. 2 for *all* physical systems, because of the finite propagation velocity of all signals. In the following, we will therefore make no use of specific properties of neural systems and, instead, study eq. 4 as a general model of a variety of systems.

We specialize in this report to the case of nearest-neighbor interactions, i.e., the sum runs over all nearest neighbors of oscillator i. We also assumed $\omega_i = \omega_0$ for all i, although most of our results do not depend on this assumption (see below). In order to achieve larger generality, we added a Gaussian noise-term (or temperature T) in the usual way:

$$\begin{aligned} \frac{d\phi_i(t)}{dt} &= \omega_0 + K \sum_j \sin(\phi_j(t-\tau) - \phi_i(t)) + \eta_i(t) \\ < \eta_i(t) > &= 0 \\ < \eta_i(t)\eta_j(t') > &= 2T\delta(i,j)\delta(t-t') \end{aligned} \tag{4}$$

where $\delta(t)$ is the Dirac delta “function”, defined by:

$$\begin{aligned} \delta(t) &= 0 \ \text{ for all } t \neq 0 \\ \int_{-\infty}^{\infty} \delta(t)dt &= 1 \end{aligned} \tag{5}$$

We simulated this system for $N = 16,384$ (128×128) oscillators and for $\omega_i = \omega_0$ for all i on the CM-2 Connection Machine, using a two-dimensional square lattice with periodic boundary conditions. Figure 1 shows the system exhibits a strong “frequency suppression:” Very rapidly, the average frequency

$$\Omega = \frac{1}{N} \sum_i < \dot{\phi}_i > \tag{6}$$

(the brackets denote the average over the random noise $\eta_i(t)$ and the dot the derivative with respect to time) of the system drops from the mean intrinsic frequency to a smaller value. This value depends on the coupling constant and the delay. With increasing coupling and delay, the mean frequency is depressed more and more, as can be seen in Figure 2 (as a function of τ). We obtain a similar frequency depression if instead of a single frequency ω_0 the oscillators have different intrinsic frequencies with a distribution that has a mean ω_0 and a width comparable to ω_0 (data not shown).

This effect can be understood in a mean-field like theory. Consider a situation where all angles change with the same frequency Ω:

$$\phi_i = \Omega t + \alpha \text{ for all } i \tag{7}$$

For $T = 0$, Ω is then determined by the self-consistency relation

$$\Omega = \omega_0 - Kn \sin(\Omega\tau) \tag{8}$$

where n is the number of neighbors (4 in the case of a square lattice with nearest-neighbor interaction). Depending on the values of its parameters, this equation has one or more than one solutions. In Fig. 1 we plot the lowest stable frequency

$$\Omega_{min} \approx \omega_0/(1 + Kn\tau) \tag{9}$$

and obtain good agreement with our simulations.

How about the other solutions? We have shown numerically [15] that, by a careful choice of the initial conditions, the system can be prepared in a way that it will enter a state with $\Omega > \Omega_{min}$. The system can be driven out of this state by thermal fluctuations. We have also presented an analytically solvable model of two delay-coupled oscillators which elucidates this behavior [15]. In this report we present a related model which also shows the observed metastable states and which involves only one oscillator.

In order to derive this model, let us consider eq. 4 for one oscillator, i.e. $N = 1$. We drop the subscript and obtain for $T = 0$:

$$\frac{d\phi(t)}{dt} = \omega_0 + K \sin(\phi(t - \tau) - \phi(t)) \tag{10}$$

Using eq. 5, this can be rewritten as

$$\frac{d\phi(t)}{dt} = \omega_0 + K \int_{-\infty}^{\infty} \delta(\tau - \tau')\ sin(\phi(t - \tau') - \phi(t))\ d\tau' \tag{11}$$

Replacing $\delta(\tau' - \tau)$ by a decaying exponential with the characteristic time τ, it is then easy to see that eq. 11 is equivalent to the following system of equations:

$$\frac{d\phi(t)}{dt} = \Omega - K sin(\phi(t) - \psi(t)/\tau) \tag{12}$$

$$\frac{d\psi(t)}{dt} = \phi(t) - \psi(t)/\tau \tag{13}$$

We have chosen the initial conditions as $\psi(0) = 0$. Taking the difference of these equations and defining

$$x = \phi - \psi/\tau \tag{14}$$

we find that x is determined by a potential function V:

$$\frac{dx}{dt} = -\frac{dV}{dx} \tag{15}$$

$$V(x) = -\Omega\ x + \frac{1}{2\tau}x^2 - K \cos x \tag{16}$$

Depending on the values of K and τ, equation 16 has a finite number (≥ 1) of stationary solutions. For sufficiently small delay or coupling constants, only the solution given in eq. 9 is obtained.

The drastic decrease of the synchronization frequency (which occurs even for small delays if the coupling strength is large) should be observed in a wide range of dynamic systems. The existence and the functional form of the potential $V(x)$ depend on the details of the coupling between the oscillators [10], but the existence of the frequency depression does not. We believe that the investigation of the nature of metastable states in dynamic systems with internal delay times which are abundant in nature is a promising direction of further research.

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Figure Legends

Figure1

The figure shows the depression of the average frequency Ω of the system of 128×128 oscillators, described by eq. 5. At $t = 0$, the average frequency Ω (defined in eq. 6) is $\omega_0 = 0.5$. Within a fraction of one oscillation period, the frequency drops to a value close to the one computed in eq. 9.

Figure2

Frequency suppression as a function of time delay τ (in units of $2\omega_0^{-1}$) for a two-dimensional array of 16,384 oscillators (diamonds) and prediction from eq. 8 (line). The average frequency Ω is plotted as a fraction of the intrinsic frequency, ω_0. Temperature, $T = 10^{-4}K$.

Figure 1: freq-depression

Figure 2: linearization